Analysis of a Curved Beam Using Classical and Shear Deformable Beam Theories

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Both the exact closed-form solution and a numerical solution by the differential quadrature method (D.Q.M.) are obtained to predict the out-of-plane static behavior of a curved beam subjected to torque, based on the curved-beam version of the classical (Bernoulli-Euler) and shear deformable (Bresse-Timoshenko) beam theories. Deflections, twist angles, angles of rotation, bending moments and twisting moments are calculated for the case of a circular arc of circular cross section with clamped and simply supported boundary conditions, and the results obtained by both methods (exact and D. Q. M.) are compared. It is found that the D.Q.M. gives good accuracy for only a limited number of grid points.

Key Words: Bending Moments, Classical Beam Theory, Curved Beam, Deflections, Differential Quadrature Method, Shear Deformable Beam Theory, Torque, Twist Angles, Twisting Moments

1. Introduction

The out-of-plane behavior of a curved shaft due to torque has been previously treated by Eubanks (1963), Cheney (1965) and Bert (1989) based on the classical curved beam theory in which transverse shear deformation is not considered. The classical theory of the bending and twisting of thin rods was utilized by Eubanks (1963) to determine the nonlinear deformation of an inextensible thin rod of circular centerline which is subjected to an axial torque. It was assumed that the rod is supported at its ends by flexible moment-free bearings, and that the ends, while constrained to lie on a given line, were free to move along this line. Cheney (1965) gave an alternative solution which would clarify the results obtained by Eubanks (1963). Eubanks's solntion was based on an unwieldy approach using the Kirchhoff rod equations, cheney's solution used thin-ring theory but obtained an erroneous solution, and Bert's solution was formulated directly from thin-ring theory and was solved directly assuming that the twisting moment was uniformly distributed.

The purpose of the present work is to obtain exact solutions for out-of-plane deflections, twist angles, angles of rotation, bending moments and twisting moments in a curved beam due to torque based on the classical and shear deformable beam theories, and to demonstrate the application of the differential quadrature method (D.Q.M.) to obtain accurate approximate solutions. The exact solutions are compared with those obtained by the D.Q.M. for the case of a circular arc of circular cross section with clamped and simply supported boundary conditions.

2. Theoretical Consideration and Closed-Form Solutions

2.1 Classical beam theory

The curved shaft considered is shown in Fig. 1. The equilibrium equations for out-of-plane bending and twisting of a thin circular arc can be expressed as follows (Volterra, 1952):

$$M_x'' + M_z' = 0\ ; \qquad -M_x + M_z' = 0 \qquad (1)$$

where M_x and M_z are the respective bending and

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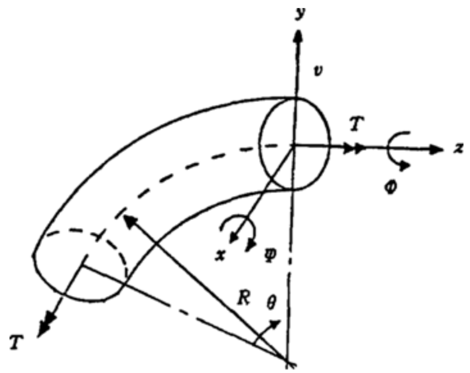

Fig. 1 Geometry of curved beam.

twisting moments at a given circumferential angular position ψ, and a prime denotes differentiation with respect to ψ.

The constitutive equations for small deflections and rotations are

$$M_x=\left(\frac{EI_x}{R}\right)\left(\Phi-\frac{v''}{R}\right);$$
$$M_z=\left(\frac{GJ}{R}\right)\left(\Phi'+\frac{v'}{R}\right) \quad (2)$$

where EI_x and GJ are the respective flexural and torsional rigidities, R is the center-line radius of the member, v is the out-of-plane deflection, and Φ is the twist angle. Substituting M_x and M_z from Eqs. (2) into Eqs. (1) gives the following governing differential equations:

$$-\frac{v''''}{R}+\frac{GJ}{EI_x}\frac{v''}{R}+\left(1+\frac{GJ}{EI_x}\right)\Phi''=0 \quad (3)$$

$$\left(1+\frac{GJ}{EI_x}\right)\frac{v''}{R}+\frac{GJ}{EI_x}\Phi''-\Phi=0 \quad (4)$$

Now, using v'' from Eq. (4) in Eq. (3), one obtains the following differential equation

$$\Phi''''+2\Phi''+\Phi=0 \quad (5)$$

which has the general solution

$$\Phi(\psi)=C_1\cos\psi+C_2\sin\psi$$
$$+C_3\psi\sin\psi+C_4\psi\cos\psi \quad (6)$$

where the C's are constants of integration. In view of Eq. (4), the general solution for the out-of-plane deflection is

$$\frac{v(\psi)}{R}=-C_1\cos\psi-C_2\sin\psi-C_3\psi\sin\psi$$
$$-C_4\psi\cos\psi-\frac{2C_3\cos\psi}{(GJ/EI_x)+1}$$
$$+\frac{2C_4\sin\psi}{(GJ/EI_x)+1}+B_0\psi+B_1 \quad (7)$$

Choosing the origin for ψ to be at the midpoint of the member and using the *antisymmetric* nature of the problem, one may write

$$v(\psi)=-v(-\psi) \quad (8)$$

Thus, B_1, C_1 and C_3 must vanish.

If the member is simply supported flexurally at each end, then the boundary conditions can be expressed in the following form

$$M_x(\pm\alpha)=0,\quad v(\pm\alpha)=0,$$
$$M_z(\pm\alpha)=\pm T \quad (9)$$

where α is one half of the total included angle of the member and T is the applied torque at each end of the member. Thus,

$$B_0=\frac{TR}{GJ},\quad C_2=\frac{\alpha}{\sin\alpha}\left(\frac{TR}{GJ}\right),\quad C_4=0 \quad (10)$$

If the member is clamped flexurally at each end, then the boundary conditions can be expressed in the following form

$$v(\pm\alpha)=0,\quad v'(\pm\alpha)=0,$$
$$M_z(\pm\alpha)=\pm T \quad (11)$$

Thus,

$$B_0=\frac{TR}{GJ}\frac{(GJ/EI_x)+1}{2q\cos\alpha+(GJ/EI_x)+1},$$
$$C_4=qB_0 \quad (12)$$
$$C_2=\frac{1}{\sin\alpha}\left[C_4\left(\frac{2\sin\alpha}{(GJ/EI_x)+1}-\alpha\cos\alpha\right)\right.$$
$$\left.+B_0\alpha\right] \quad (13)$$

where $q=(\sin\alpha-\alpha\cos\alpha)/(\sin\alpha\cos\alpha-\alpha)$.

2.2 Shear deformable beam theory

Rao(1971), neglecting the warping deformation (as is appropriate for the circular cross section considered here), obtained the following equilibrium equations:

$$\frac{v''}{R}-\Psi'=0 \quad (14)$$

$$-\chi\frac{G}{E}s\frac{v'}{R}+(1+\mu)\Phi'-\Psi''$$
$$+\left(\mu+\chi\frac{G}{E}s\right)\Psi=0 \quad (15)$$

$$\mu\Phi''-\Phi+(1+\mu)\Psi'=0 \quad (16)$$

Hhere Ψ is the angle of rotation due to pure out-of-plane bending, and χ is the shear correction factor. For simplicity of analysis, the following dimensionless variables have been introduced:

$$s=AR^2/I_x,\quad \mu=GJ/EI_x \tag{17}$$

where A is the cross-sectional area, s is the slenderness ratio, and μ is the rigidity ratio of the member. Substituting v' from Eq. (15) into Eq. (14) and using the antisymmetric nature of the problem give the following general solutions

$$\Phi(\phi)=C_1\sin\phi+C_2\phi\cos\phi \tag{18}$$

$$\frac{v(\phi)}{R}=-C_1\sin\phi-C_2\phi\cos\phi+\frac{2C_2\sin\phi}{\mu+1}+C_3\phi \tag{19}$$

$$\Psi(\phi)=-C_1\cos\phi+C_2\left(\phi\sin\phi+\cos\phi-\frac{2\mu}{\mu+1}\cos\phi\right)+C_4 \tag{20}$$

If the member is simply supported flexurally at each end, then the boundary conditions can be expressed in the following form:

$$M_x(\pm\alpha)=\pm\frac{EI_x}{R}(\Phi-\Psi')=0,$$
$$v(\pm\alpha)=0,$$
$$M_z(\pm\alpha)=\pm\frac{GJ}{R}(\Psi+\Phi')=\pm T \tag{21}$$

Thus

$$C_4=\frac{TR}{GJ},\quad C_3=\left(1+\frac{J}{I_x\chi s}\right)C_4,$$
$$C_1=\frac{\alpha}{\sin\alpha}\left(1+\frac{J}{I_x\chi s}\right)C_4,\quad C_2=0 \tag{22}$$

If the member is clamped flexurally at each end, then the boundary conditions can be expressed in the following form

$$\Psi(\pm\alpha)=0,\quad v(\pm\alpha)=0,$$
$$M_z(\pm\alpha)=\pm\frac{GJ}{R}(\Psi+\Phi')=\pm T \tag{23}$$

Thus,

$$C_4=\frac{TR}{GJ}\frac{\mu+1}{2p\cos\alpha+\mu+1},$$
$$C_3=\left(1+\frac{J}{I_x\chi s}\right)C_4$$
$$C_1=\frac{1}{\sin\alpha}\left[C_2\left(\frac{2\sin\alpha}{\mu+1}-\alpha\cos\alpha\right)+C_3\alpha\right],$$
$$C_2=pC_4 \tag{24}$$

where $p=[\sin\alpha-(1+J/I_x\chi s)\alpha\cos\alpha]/(\sin\alpha\cos\alpha-\alpha)$.

3. Differential Quadrature Method

In many cases, moderately accurate solutions which can be calculated rapidly are desired at a few points in the respective physical domains. These solutions have traditionally been obtained by the standard finite difference and finite element methods which must be computed based on a large number of points. The mentioned methods depend strongly on the nature and refinement of the discretization of the domain. However, in order to get results even with only limited accuracy at or near a point of interest for a complicated problem, solutions often have to be computed based on a large number of surrounding points since the accuracy and stability of the aforementioned classical methods depend strongly on the nature and refinement scheme adopted to discretize the domain. Consequently, computational efforts are often considerable for these standard methods. In order to overcome the aforementioned complexities, an efficient procedure called the differential quadrature method was introduced by Bellman and Casti (1971). By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to the static analysis of structural components by Jang, Bert and Striz (1989). The versatility of the D.Q.M. to engineering analysis in general, and to structural analysis in particular, is becoming increasingly evident by the number of related publications in recent years. Kukreti, Farsa and Bert (1992) calculated the fundamental frequencies of tapered plates, and Farsa, Kukreti and Bert (1993) applied the method to the analysis and detailed parametric evaluation of the fundamental frequencies of general anisotropic and laminated plates. In another development, the quadrature method was introduced in lubrication mechanics by Malik and Bert (1994). Kang, Bert and Striz (1995) applied the method to the vibration analysis of shear deformable circular arches. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i=\sum_{j=1}^{N} W_{ij}f(x_j) \quad \text{for } i, j=1, 2, \cdots, N \tag{25}$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, expresses as the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$$f_k(x)=x^{k-1} \text{ for } k=1, 2, 3, \cdots, N \tag{26}$$

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij}x_j^{k-1}=(k-1)(k-2)\cdots(k-n)x_i^{k-n-1} \quad \text{for } i, k=1, 2, \cdots, N \tag{27}$$

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, W_{ij}, since the coefficient matrix is a Vandermonde matrix, which always has an inverse as described by Hamming (1973). Thus, the method can be used to express the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

4. Application

4.1 Classical beam theory

Applying the differential quadrature method to Eqs. (3) and (4) gives

$$-\frac{1}{R\theta_0^4}\sum_{j=1}^{N} D_{ij}v_j+\frac{GJ}{EI_x}\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{ij}v_j+\left(1+\frac{GJ}{EI_x}\right)\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\Phi_j=0 \tag{28}$$

$$\left(1+\frac{GJ}{EI_x}\right)\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{ij}v_j+\frac{GJ}{EI_x}\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\Phi_j-\Phi_i=0 \tag{29}$$

where B_{ij} and D_{ij} are the weighting coefficients for the second and fourth derivatives, respectively, along the dimensionless axis X defined as

$$X=\frac{\theta}{\theta_0} \tag{30}$$

Here, θ is the circumferential angular position measured from the left support and $\theta_0(=2\alpha)$ is the total opening angle.

Considering the symmetry of the loading, one can express the boundary conditions for simply supported ends, given by Eqs. (9), and the deflection at the midpoint of the member in the differential quadrature form as follows:

$$v_1=0 \quad \text{at} \quad X=0 \tag{31}$$

$$\frac{1}{\theta_0}\left(\sum_{j=1}^{N} A_{2j}\Phi_j+\frac{1}{R}\sum_{j=1}^{N} A_{2j}v_j\right)=\frac{TR}{GJ} \quad \text{at } X=0+\delta \tag{32}$$

$$\frac{EI_x}{R}\left(\Phi_2-\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{2j}v_j\right)=0 \quad \text{at } X=0+\delta \tag{33}$$

$$v_{(N+1)/2} \quad \text{at } X=0.5 \tag{34}$$

$$\frac{EI_x}{R}\left(\Phi_{N-1}-\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{(N-1)j}v_j\right)=0 \quad \text{at } X=1-\delta \tag{35}$$

$$\frac{1}{\theta_0}\left(\sum_{j=1}^{N} A_{(N-1)j}\Phi_j+\frac{1}{R}\sum_{j=1}^{N} A_{(N-1)j}v_j\right)=\frac{TR}{GJ} \quad \text{at } X=1-\delta \tag{36}$$

$$v_N=0 \quad \text{at } X=0 \tag{37}$$

where A_{2j} and $A_{(N-1)}$ are the weighting coefficients for the first derivatives. Here δ denotes a very small dimensionless distance measured along the dimensionless axis from each boundary end. This set of equations together with the appropriate boundary conditions can be solved for the deflection and twist angle.

Similarly, the boundary conditions for clamped ends, given by Eqs. (11), and the deflection at the midpoint of the member can be expressed in the differential quadrature form as follows:

$$v_1=0 \quad \text{at } X=0 \tag{38}$$

$$\frac{1}{\theta_0}\left(\sum_{j=1}^{N} A_{2j}\Phi_j+\frac{1}{R}\sum_{j=1}^{N} A_{2j}v_j\right)=\frac{TR}{GJ}$$

$$\text{at } X=0+\delta \tag{39}$$

$$\sum_{j=1}^{N} A_{2j} v_j = 0 \quad \text{at } X=0+\delta \tag{40}$$

$$v_{(N+1)/2} \quad \text{at } X=0.5 \tag{41}$$

$$\sum_{j=1}^{N} A_{(N-1)j} v_j = 0 \quad \text{at } X=1-\delta \tag{42}$$

$$\frac{1}{\theta_0}\left(\sum_{j=1}^{N} A_{(N-1)j}\Phi_j + \frac{1}{R}\sum_{j=1}^{N} A_{(N-1)j} v_j\right) = \frac{TR}{GJ}$$

$$\text{at } X=1-\delta \tag{43}$$

$$v_N = 0 \quad \text{at } X=0 \tag{44}$$

The bending moments and twisting moments, given by Eq. (2), can be expressed in differential quadrature form as follows:

$$M_x = \left(\frac{EI_x}{R}\right)\left(\Phi_i - \frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{ij} v_j\right) \tag{45}$$

$$M_z = \left(\frac{GJ}{R\theta_0}\right)\left(\sum_{j=1}^{N} A_{ij}\Phi_j + \frac{1}{R}\sum_{j=1}^{N} A_{ij} v_j\right) \tag{46}$$

This set of equations together with the appropriate boundary conditions can be solved to obtain the out-of-plane static behavior of the curved beam subjected to end torques.

4.2 Shear deformable beam theory

Laura and Gutierrez (1993) applied the differential quadrature method to the analysis of vibrating Bresse-Timoshenko straight beams. Applying the method to shear-deformable curved beams, Eqs. (14), (15), and (16), one obtains

$$\kappa\frac{G}{E}s\frac{1}{R\theta_0^2}\sum_{j=1}^{N} B_{ij} v_j - \kappa\frac{G}{E}s\frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Psi_j = 0 \tag{47}$$

$$-\kappa\frac{G}{E}s\frac{1}{R\theta_0}\sum_{j=1}^{N} A_{ij} v_j + (1+\mu)\frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Phi_j - \frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\Psi_j + \left(\mu + \kappa\frac{G}{E}s\right)\Psi_i = 0 \tag{48}$$

$$\mu\frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\Phi_j - \Phi_i + (1+\mu)\frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Psi_j = 0 \tag{49}$$

The boundary conditions for simply supported ends, given by Eqs. (21), and the deflection at the midpoint of the member can be expressed in differential quadrature form as follows:

$$v_1 = 0 \quad \text{at } X=0 \tag{50}$$

$$\frac{1}{\theta_0}\sum_{j=1}^{N} A_{2j}\Phi_j + \Psi_2 = TR/GJ$$

$$\text{at } X=0+\delta \tag{51}$$

$$\frac{EI_x}{R}\left(\Phi_2 - \frac{1}{\theta_0}\sum_{j=1}^{N} A_{2j}\Psi_j\right) = 0$$

$$\text{at } X=0+\delta \tag{52}$$

$$v_{(N+1)/2} = 0 \quad \text{at } X=0.5 \tag{53}$$

$$\frac{EI_x}{R}\left(\Phi_{(N-1)} - \frac{1}{\theta_0}\sum_{j=1}^{N} A_{(N-1)j}\Psi_j\right) = 0$$

$$\text{at } X=1-\delta \tag{54}$$

$$\frac{1}{\theta_0}\sum_{j=1}^{N} A_{(N-1)j}\Phi_j + \Psi_{(N-1)} = TR/GJ$$

$$\text{at } X=1-\delta \tag{55}$$

$$v_N = 0 \quad \text{at } X=1 \tag{56}$$

Similarly, the boundary conditions for clamped ends, given by Eqs (23), and the deflection at the midpoint of the member can be expressed in differential quadrature form as follows:

$$v_1 = 0 \quad \text{at } X=0 \tag{57}$$

$$\frac{1}{\theta_0}\sum_{j=1}^{N} A_{2j}\Phi_j + \Psi_2 = TR/GJ \quad \text{at } X=0+\delta \tag{58}$$

$$\Psi_2 = 0 \quad \text{at } X=0+\delta \tag{59}$$

$$v_{(N+1)/2} = 0 \quad \text{at } X=0.5 \tag{60}$$

$$\Psi_{(N-1)} = 0 \quad \text{at } X=1-\delta \tag{61}$$

$$\frac{1}{\theta_0}\sum_{j=1}^{N} A_{(N-1)j}\Phi_j + \Psi_{(N-1)} = TR/GJ$$

$$\text{at } X=1-\delta \tag{62}$$

$$v_N = 0 \quad \text{at } X=1 \tag{63}$$

The bending moments and twisting moments can be expressed in differential quadrature form as follows:

$$M_x = \left(\frac{EI_x}{R}\right)\left(\Phi_i - \frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Psi_j\right),$$

$$M_z = \left(\frac{GJ}{R}\right)\left(\Psi_i + \frac{1}{\theta_0}\sum_{j=1}^{N} A_{ij}\Phi_j\right) \tag{64}$$

5. Numerical Results and Comparisons

Based on the above derivations, the deflections, twist angles, angles of rotation, bending moments and twisting moments for the out-of-plane behavior of the member are calculated by a closed-form solution and by the differential quadrature method. The deflections, twist angles, angles of rotation, bending moments and twisting moments are evaluated for the case of a circular arc with circular cross section under clamped and simply supported boundary conditions, and numerical results are compared between the two solution methods. The ratio of the center-line radius R to the radius of cross section r is 5.0, and the

Table 1 Twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section for a range of grid points using classical beam theory; $\nu=0.3$, $\theta_0=180°$ and $\delta = 1\times10^{-5}$.

Exact θ, degrees	Number of grid points, N			
0°	7	9	11	13
−0.8511	−0.8597	−0.8509	−0.8512	−0.8512

Table 2 Twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section for a range of δ using classical beam theory; $\nu=0.3$, $\theta_0=180°$ and $N=13$.

Exact θ, degrees	δ			
0°	1×10^{-3}	1×10^{-4}	1×10^{-5}	1×10^{-6}
−0.8511	−0.8537	−0.8514	−0.8512	−0.8511

Table 3 Deflection $v^* = vGJ/TR^2$ and twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section using classical beam theory; $\nu=0.3$ and $\theta_0=180°$.

θ, degrees	v^*		Φ^*	
	Exact	D.Q.M.	Exact	D.Q.M.
0°	0.0	0.0	−0.8511	−0.8512
18°	−0.009935	−0.009933	−0.5623	−0.5623
36°	−0.02435	−0.02435	−0.3359	−0.3359
54°	−0.02863	−0.02863	−0.1767	−0.1767
72°	−0.01897	−0.01897	−0.07281	−0.07281
90°	0.0	0.0	0.0	-2.0×10^{-7}

Table 4 Deflection $v^* = vGJ/TR^2$ and twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section using classical beam theory; $\nu=0.3$ and $\theta_0=90°$.

θ, degrees	v^*		Φ^*	
	Exact	D.Q.M.	Exact	D.Q.M.
0°	0.0	0.0	−0.6268	−0.6268
9°	−0.0004702	−0.0004701	−0.4779	−0.4779
18°	−0.001124	−0.001124	−0.3439	−0.3439
27°	−0.001298	−0.001298	−0.2221	−0.2221
36°	−0.0008508	−0.0008507	−0.1088	−0.1088
45°	0.0	0.0	0.0	-2.0×10^{-7}

Table 5 Deflection $v^*=vGJ/TR^2$ and twist angle $\Phi^*=\Phi GJ/TR$ for out-of-plane behavior of circular arc beam and simply supported ends with circular cross section using classical beam theory; $\nu=0.3$ and $\theta^\circ=180°$.

θ, degrees	v^*		Φ^*	
	Exact	D.Q.M.	Exact	D.Q.M.
0°	0.0	0.0	−1.571	−1.571
18°	0.2373	0.2373	−1.494	−1.494
36°	0.3283	0.3283	−1.271	−1.271
54°	0.2950	0.2950	−0.9233	−0.9233
72°	0.1712	0.1712	−0.4854	−0.4854
90°	0.0	0.0	0.0	-5.5×10^{-7}

Table 6 Deflection $v^*=vGJ/TR^2$ and twist angle $\Phi^*=\Phi GJ/TR$ for out-of-plane behavior of circular arc beam and simply supported ends with circular cross section using classical beam theory; $\nu=0.3$ and $\theta_0=90°$.

θ, degrees	v^*		Φ^*	
	Exact	D.Q.M.	Exact	D.Q.M.
0°	0.0	0.0	−0.7854	−0.7854
9°	−0.02455	−0.02455	−0.6529	−0.6529
18°	−0.03302	−0.03302	−0.5043	−0.5043
27°	−0.02907	−0.02907	−0.3432	−0.3432
36°	−0.01668	−0.01668	−0.1738	−0.1738
45°	0.0	0.0	0.0	-2.0×10^{-7}

Poisson's ratio of the member, ν, is 0.3. The shear for a circular cross section using elasticity theory.

Table 7 Bending moment $M_x^*=M_x/T$ and twisting moment $M_z^*=M_z/T$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section using classical beam theory; $\nu=0.3$ and $\theta_0=180°$.

θ, degrees	M_x^*		M_z^*	
	Exact	D.Q.M.	Exact	D.Q.M.
0°	−0.7197	−0.7197	1.0	1.0
18°	−0.6844	−0.6844	0.7776	0.7776
36°	−0.5822	−0.5822	0.5770	0.5770
54°	−0.4230	−0.4230	0.4178	0.4178
72°	−0.2224	−0.2224	0.3156	0.3156
90°	0.0	0.0	0.2803	0.2804

correction factor χ is the established value (0.89)

Table 8 Bending moment $M_x^*=M_x/T$ and twisting moment $M_z^*=M_z/T$ for out-of-plane behavior of circular arc beam and simply supported ends with circular cross section using classical beam theory; $\nu=0.3$ and $\theta_0=180°$.

θ,	M_x^*		M_z^*	
degrees	Exact	D.Q.M.	Exact	D.Q.M.
0°	0.0	0.0	1.0	1.0
18°	0.0	0.0	1.0	1.0
36°	0.0	0.0	1.0	1.0
54°	0.0	0.0	1.0	1.0
72°	0.0	0.0	1.0	1.0
90°	0.0	0.0	1.0	1.0

Table 9 Deflection $v^*=vGJ/TR^2$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta_0=90°$ and $\theta_0=180°$.

θ,	v^*		θ,	v^*	
($\theta_0=90°$)	Exact	D.Q.M.	($\theta_0=180°$)	Exact	D.Q.M.
0°	0.0	0.0	0°	0.0	0.0
9°	0.003807	0.003808	18°	−0.004603	−0.004600
18°	0.004696	0.004697	36°	−0.01697	−0.01698
27°	0.003868	0.003868	54°	−0.02201	−0.02200
36°	0.002127	0.002127	72°	−0.01512	−0.01512
45°	0.0	0.0	90°	0.0	0.0

Table 10 Twist angle $\Phi^*=\Phi GJ/TR$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta_0=90°$ and $\theta_0=180°$.

θ,	Φ^*		θ,	Φ^*	
($\theta_0=90°$)	Exact	D.Q.M.	($\theta_0=180°$)	Exact	D.Q.M.
0°	−0.6771	−0.6771	0°	−0.8864	−0.8865
9°	−0.5255	−0.5255	18°	−0.5958	−0.5959
18°	−0.3841	−0.3841	36°	−0.3645	−0.3645
27°	−0.2510	−0.2510	54°	−0.1974	−0.1974
36°	−0.1240	−0.1240	72°	−0.08371	−0.08372
45°	0.0	-2.9×10^{-7}	90°	0.0	-5.5×10^{-7}

Table 11 Angle of rotation $\Psi^* = \Psi GJ/TR$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta_0=90°$ and $\theta_0=180°$.

θ, ($\theta_0=90°$)	Ψ^*		θ, ($\theta_0=180°$)	Ψ^*	
	Exact	D.Q.M.		Exact	D.Q.M.
0°	0.0	0.0	0°	0.0	0.0
9°	−0.02341	−0.02341	18°	−0.06043	−0.06043
18°	−0.03769	−0.03769	36°	−0.05525	−0.05525
27°	−0.04570	−0.04570	54°	−0.01945	−0.01944
36°	−0.04961	−0.04961	72°	0.01617	0.01617
45°	−0.05076	−0.05076	90°	0.03053	0.03052

Table 12 Deflection $v^* = vGJ/TR^2$ for out-of-plane behavior of circular arc beam and simply supported ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta_0=90°$ and $\theta_0=180°$.

θ, ($\theta_0=90°$)	v^*		θ, ($\theta_0=180°$)	v^*	
	Exact	D.Q.M.		Exact	D.Q.M.
0°	0.0	0.0	0°	0.0	0.0
9°	0.02510	0.02510	18°	0.2426	0.2426
18°	0.03376	0.03376	36°	0.3357	0.3357
27°	0.02973	0.02973	54°	0.3016	0.3016
36°	0.01705	0.01705	72°	0.1751	0.1751
45°	0.0	0.0	90°	0.0	0.0

Table 13 Twist angle $\Phi^* = \Phi GJ/TR$ for out-of-plane behavior of circular arc beam and simply supported ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta_0=90°$ and $\theta_0=180°$.

θ, ($\theta_0=90°$)	Φ^*		θ, ($\theta_0=180°$)	Φ^*	
	Exact	D.Q.M.		Exact	D.Q.M.
0°	−0.8035	−0.8031	0°	−1.6061	−1.6061
9°	−0.6675	−0.6675	18°	−1.5275	−1.5275
18°	−0.5156	−0.5156	36°	−1.2994	−1.2994
27°	−0.3510	−0.3510	54°	−0.9440	−0.9440
36°	−0.1777	−0.1777	72°	−0.4963	−0.4963
45°	0.0	-7.3×10^{-7}	90°	0.0	-2.9×10^{-7}

Table 14 Angle of rotation $\Psi^* = \Psi GJ/TR$ for out-of-plane behavior of circular arc beam and simply supported ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta_0=90°$ and $\theta_0=180°$.

θ,	Ψ^*		θ,	Ψ^*	
($\theta_0=90°$)	Exact	D.Q.M.	($\theta_0=180°$)	Exact	D.Q.M.
0°	0.1970	0.1970	0°	1.0	1.0
9°	0.08122	0.08122	18°	0.5037	0.5037
18°	−0.01190	−0.01190	36°	0.05596	0.05596
27°	−0.08010	−0.08010	54°	−0.2994	−0.2994
36°	−0.1217	−0.1217	72°	−0.5275	−0.5275
45°	−0.1357	−0.1357	90°	−0.6061	−0.6061

Table 15 Bending moment $M_x^* = M_x/T$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta_0=90°$ and $\theta_0=180°$.

θ,	M_x^*		θ,	M_x^*	
($\theta_0=90°$)	Exact	D.Q.M.	($\theta_0=180°$)	Exact	D.Q.M.
0°	−0.6395	−0.6395	0°	−0.7196	−0.7197
9°	−0.5316	−0.5316	18°	−0.6844	−0.6844
18°	−0.4106	−0.4106	36°	−0.5822	−0.5822
27°	−0.2795	−0.2795	54°	−0.4230	−0.4230
36°	−0.1415	−0.1415	72°	−0.2224	−0.2224
45°	0.0	-4.0×10^{-5}	90°	0.0	-2.7×10^{-7}

Table 16 Bending moment $M_x^* = M_x/T$ for out-of-plane behavior of circular arc beam and simply supported ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta_0=90°$ and $\theta_0=180°$.

θ,	M_x^*		θ,	M_x^*	
($\theta_0=90°$)	Exact	D.Q.M.	($\theta_0=180°$)	Exact	D.Q.M.
0°	0.0	0.0	0°	0.0	0.0
9°	0.0	0.0	18°	0.0	0.0
18°	0.0	0.0	36°	0.0	0.0
27°	0.0	0.0	54°	0.0	0.0
36°	0.0	0.0	72°	0.0	0.0
45°	0.0	0.0	90°	0.0	0.0

Table 17 Twisting moment $M_z^* = M_z / T$ for out-of-plane behavior of circular arc beam and clamped ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta_0=90°$ and $\theta_0=180°$.

θ, ($\theta_0=90°$)	M_z^* Exact	M_z^* D.Q.M.	θ, ($\theta_0=180°$)	M_z^* Exact	M_z^* D.Q.M.
0°	1.0	1.0	0°	1.0	1.0
9°	0.9078	0.9078	18°	0.7776	0.7776
18°	0.8337	0.8337	36°	0.5770	0.5770
27°	0.7794	0.7794	54°	0.4178	0.4178
36°	0.7463	0.7463	72°	0.3156	0.3156
45°	0.7351	0.7351	90°	0.2803	0.2804

Table 18 Twisting moment $M_z^* = M_z / T$ for out-of-plane behavior of circular arc beam and simply supported ends with circular cross section using shear deformable beam theory; $\nu=0.3$, $R/r=5.0$, $\theta°=90°$ and $\theta°=180°$.

θ, ($\theta_0=90°$)	M_z^* Exact	M_z^* D.Q.M.	θ, ($\theta_0=180°$)	M_z^* Exact	M_z^* D.Q.M.
0°	1.0	1.0	0°	1.0	1.0
9°	1.0	1.0	18°	1.0	1.0
18°	1.0	1.0	36°	1.0	1.0
27°	1.0	1.0	54°	1.0	1.0
36°	1.0	1.0	72°	1.0	1.0
45°	1.0	1.0	90°	1.0	1.0

Table 19 Deflection $v^* = vGJ/TR^2$ for variations in ratio of out-of-plane behavior of circular arc beam and simply supported ends with circular cross section using shear deformable beam theory; $\nu=0.3$ and $\theta_0=180°$.

θ, degrees	v^* (Exact)						
	Ratio of center-line to radius of cross section, R/r						
$\theta_0=180°$	1.0	5.0	10.0	15.0	20.0	25.0	50.0
0°	0.0	0.0	0.0	0.0	0.0	0.0	0.0
18°	0.3706	0.2426	0.2386	0.2379	0.2376	0.2375	0.2373
36°	0.5128	0.3357	0.3302	0.3291	0.3288	0.3286	0.3284
54°	0.4607	0.3016	0.2966	0.2957	0.2954	0.2952	0.2950
72°	0.2675	0.1751	0.1722	0.1718	0.1715	0.1714	0.1713
90°	0.0	0.0	0.0	0.0	0.0	0.0	0.0

For classical thin curved beam theory, Tables 1 and 2 present the results of convergence studies relative to the number of grid points N δ parameter, respectively. Table 1 shows that the accuracy of the numerical solution increases with increasing N, passes through a maximum, but the decreases due to numerical instabilities if N becomes too large. Table 2 shows how the numerical solution is sensitive to the choice of δ. The optimal value of δ is found to be 1×10^{-5} to 1×10^{-6}, which is obtained from trial-and-error calculations. The solution accuracy decreases due to numerical instabilities if δ becomes too small or too large. The remainder of the numerical results are computed with thirteen discrete points along the dimensionless x-axis and $\delta=1\times10^{-5}$. For members with either clamped or simply supported ends and opening angles of 180° and 90°, the deflections v^* and the twist angles Φ^* are summarized for classical beam theory in Tables 3～6, and the bending moments M_x^* and twisting moments M_z^* are summarized for classical beam theory in Tables 7 and 8. The deflections v^*, the twist angles Φ^* and angles of rotation Ψ^* due to pure bending are summarized for shear deformable beam theory in Tables 9～14, and the bending moments M_x^* and twisting moments M_z^* are summarized for shear deformable beam theory in Tables 15～18. Table 19 shows the variations in the ratio of center-line radius to radius of cross section. From Tables 3～6 and Tables 9～14, for the case of clamped ends, both deflections v^* and twist angles Φ^* are smaller than those for the case of simply supported ends for the opening angles of 90° and 180°, and for the opening angle of 90°, the angles of rotation Ψ^* are also smaller than those for the opening angle of 180^0 for clamped ends and simply supported ends. From Tables 7～8 and Tables 15～18, the twisting moment distribution M_z^* is uniformly distributed for simply supported ends, but varies for clamped ends. The bending moment M_x^* is identically zero along the entire length of the member for simply supported ends, but varies for clamped ends. From Table 19, it is seen that when the ratio R/r is less than 10.0, the variations in the ratio have a significant effect on the deflections. Table 19 also shows that shearing deformable beam theory becomes more significant as the ratio decreases be low 10.0. Eubanks (1963) calculated the deflections based on the classical curved beam theory using the Kirchhoff rod equations. The deflections determined by Eubanks (1963) for simply supported ends and opening angles of 180° were infinite displacements due to an erroneous solution. As can be seen, the numerical results show excellent agreement with the exact solutions.

6. Conclusions

Both closed-form analytical and differential quadrature methods were used to compute the deflections, twist angles, angles of rotation, bending moments and twisting moments for out-of-plane static behavior of a curved beam based on the classical and shear deformable beam theories. The D. Q. M. gives results which agree very well with the exact ones for the cases treated, while requiring only a limited number of grid points.

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